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LETTER TO THE EDITOR

Explicit formula for the multiple generating function of product of generalized Laguerre polynomials

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Abstract. Using an operator approach, Messina and Paladimo have recently demonstrated a method of evaluating a multiple generating function of the product of generalized Laguerre polynomials, and arrived at the solution which is given implicitly. By using an addition theorem of Vilenkin for the Laguerre polynomials, which has its origin in group theory, we are able to give an explicit formula for the multiple generating function in terms of elementary symmetric functions. It is further shown that the multiple generating function is a symmetric function, an interesting property which is not altogether obvious from the solution via the operator approach.

1. Introduction

Arising from works on the properties of the ground state of a two-level system linearly coupled to a set of quantum harmonic oscillators, Messina and Paladimo [1] have discussed a method of evaluating the following multiple generating function of the product of generalized Laguerre polynomials, viz

$$\sum_{s_1, s_2, \ldots, s_n = 0}^{\infty} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} L_{s_2}^{(s_1 - s_2)}(x) L_{s_3}^{(s_2 - s_3)}(x) \ldots L_{s_1}^{(s_n - s_1)}(x). \tag{1}$$

Their approach essentially relies on the observation that the trace of a certain sequence of operators can be evaluated using two different bases. By equating the two constructed expressions of the trace in a common region of convergence, they have succeeded in obtaining an expression, in implicit form, of the multiple generating function in (1).

As alluded to in [1], this generating function seems to have wide applications, and occurs in a myriad of situations in physics and chemistry where a displaced number of states of a harmonic oscillator are involved. It is therefore highly desirable that an exact expression of (1) can be found. By using a special case of an interesting addition theorem for the generalized Laguerre polynomials discussed by Vilenkin in connection with the group representation of special functions, we are able give an explicit expression for the multiple generating function in (1) in terms of elementary symmetric functions. We show further that it is a symmetric function of the variables $z_1, z_2, \ldots, z_n$, a fact which is not easily discernible from the solution by Messina and Paladimo.

2. Derivation of the explicit result

The interesting special case of Vilenkin's addition theorem, a bilinear generating function formula for the generalized Laguerre polynomials endowed with the property of additivity of the parameters [2], (cf [3]) may be stated as follows:

$$\sum_{j=0}^{\infty}(c_1)^{j-l}L_l^{(j-l)}(b_1c_1)(c_2)^{k-j}L_j^{(k-j)}(b_2c_2)$$
$$= \mathrm{e}^{-c_1b_2}(c_1+c_2)^{l-k}L_l^{(l-k)}[(b_1+b_2)(c_1+c_2)] \tag{2}$$

where the parameters c_1, c_2, b_1 and b_2 are arbitrary, and $|c_1| < |c_2|$.

In order to bring Vilenkin's formula to bear on the derivation, let us consider first of all the following related multiple generating function:

$$I_n = \sum_{s_1,s_2,\ldots,s_n=0}^{\infty}\left(\frac{c_0}{c_1}\right)^{s_1}\left(\frac{c_1}{c_2}\right)^{s_2}\ldots\left(\frac{c_{n-1}}{c_n}\right)^{s_n}L_{s_2}^{(s_1-s_2)}(b_1c_1)L_{s_3}^{(s_2-s_3)}(b_2c_2)\ldots$$
$$\ldots L_{s_1}^{(s_n-s_1)}(b_nc_n). \tag{3}$$

Let us define

$$\phi(s_i,s_{i+1}) = \left(\frac{c_{i-1}}{c_i}\right)^{s_i}L_{s_i}^{(s_i-s_{i+1})}(b_ic_i) \qquad 2\leqslant i\leqslant n-1$$
$$\phi(s_n,s_1) = \left(\frac{c_{n-1}}{c_n}\right)^{s_n}L_{s_n}^{(s_n-s_1)}(b_nc_n) \qquad i=n.$$

If we sum from (3), starting with s_2 onwards and successively up to s_{n-1}, by using the following recursion relation

$$J_k(s_1,s_{k+2}) = \sum_{s_{k+1}=0}^{\infty}\phi(s_{k+1},s_{k+2})J_{k-1}(s_1,s_{k+1}) \qquad 1\leqslant k<n-1$$
$$J_0(s_1,s_2) = L_{s_2}^{(s_1-s_2)}(b_1c_1)$$

we find, on account of Vilenkin's result in (3), that

$$\begin{aligned}J_k(s_1,s_{k+2}) &= \exp-\{c_1b_2+(c_1+c_2)b_3+\cdots+(c_1+c_2+\cdots+c_k)b_{k+1}\}\\ &\quad\times\left(\frac{c_1}{c_1+c_2+\cdots+c_{k+1}}\right)^{s_1}\left(\frac{c_1+c_2+\cdots+c_{k+1}}{c_{k+1}}\right)^{s_{k+2}}\\ &\quad\times L_{s_1}^{(s_{k+2}-s_1)}[(b_1+b_2+\cdots+b_{k+1})(c_1+c_2+\cdots+c_{k+1})]\\ &\quad 1\leqslant k<n-1.\end{aligned}$$

The last summation for s_n is

$$J_{n-1}(s_1,s_1) = \sum_{s_n=0}^{\infty}\phi(s_n,s_1)J_{n-2}(s_1,s_n)$$
$$= \exp-\left\{\sum_{j=1}^{n-1}\sum_{i=1}^{j}c_ib_{j+1}\right\}\left(\frac{c_1}{c_n}\right)^{s_1}L_{s_1}\left[\left(\sum_{i=1}^{n}b_i\right)\left(\sum_{j=1}^{n}c_j\right)\right].$$

Finally, the first summation with respect to s_1 is now performed, giving

$$I_n = \sum_{s_1=0}^{\infty}\left(\frac{c_0}{c_1}\right)^{s_1}J_{n-1}(s_1,s_1)$$
$$= (1-\alpha_n)\exp-(\alpha_n\beta_n) \qquad n\geqslant 1 \tag{4}$$

where

$$\alpha_n = \frac{c_n}{c_n-c_0} \tag{5}$$

$$\beta_n = \sum_{j=1}^{n-1}\sum_{i=1}^{j}c_ib_{j+1}+\left(\sum_{i=1}^{n}b_i\right)\left(\sum_{j=1}^{n}c_j\right) \tag{6}$$

on using the following well-known generating function for the simple Laguerre polynomials [4]:

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) \qquad |t| < 1. \tag{7}$$

If we now make the following identifications

$$\begin{aligned} &\frac{c_{i-1}}{c_i} = z_i \qquad 1 \leqslant i \leqslant n \\ &b_j c_j = x \qquad 1 \leqslant j \leqslant n \end{aligned} \tag{8}$$

we see that I_n in (3) is identical to the multiple generating function in (1) which we set out to evaluate, and furthermore from (8) we have

$$b_i c_j = \begin{cases} x(z_i z_{i+1} \ldots z_{j-1})^{-1} & i < j \\ x & i = j \\ x(z_j z_{j+1} \ldots z_{i-1}) & i > j. \end{cases} \tag{9}$$

If we use the relations in (9), it is quite straightforward, albeit a little tedious, to show that explicitly

$$\begin{aligned} &1 - \alpha_n = \frac{1}{1-\sigma_n} \\ &\alpha_n \beta_n = \frac{x}{1-\sigma_n}\left(n\sigma_n + \sum_{i=1}^{n-1} \sigma_i\right) \end{aligned} \tag{10}$$

where the elementary symmetric functions σ_i are defined as

$$\begin{aligned} &\sigma_0 = 1 \\ &\sigma_1 = \sum_{i=1}^{n} z_i \\ &\sigma_2 = \sum_{i<j}^{n} z_i z_j \\ &\sigma_3 = \sum_{i<j<k}^{n} z_i z_j z_k \\ &\vdots \\ &\sigma_n = z_1 z_2 \ldots z_n. \end{aligned}$$

Finally, with the substitutions of the values from (10) into (4), I_n gives the explicit formula for the multiple generating function relation in (1).

3. Discussions

It is easily verified that I_1, I_2 and I_3 from (4) check with the corresponding results given in (50), (51) and (52), respectively, as mentioned in [1].

Furthermore, from the following easily derived generating function relation for the σ_i, $0 \leqslant i \leqslant n$,

$$\psi(t) = \sum_{i=0}^{n} t^i \sigma_i = \prod_{i=1}^{n} (1 + z_i t) \tag{11}$$

the result in (4) may be written as

$$I_n = \frac{1}{1-\sigma_n} \exp - \left\{ \frac{x}{1-\sigma_n} [(n-1)\sigma_n + \psi(1) - 1] \right\} \qquad n \geqslant 1 \tag{12}$$

from which it is clear that I_n is a symmetric function of the variables $z_1, z_2, \ldots, z_n$. This property is not apparent from the form of (1), nor is it obvious from the implicit solution given in [1].

The convergence requirements in the successive summation operations using Vilenkin's formula in (2) and, subsequently, the generating function relation of the simple Laguerre polynomials imply *iter alia* that

$$\begin{aligned} &\left| \sum_{i=1}^{j} c_i \right| < |c_{j+1}| \qquad 1 \leqslant j \leqslant n-1 \\ &|c_0| < |c_n|. \end{aligned} \tag{13}$$

On account of the relations in (8), these are translated into

$$\begin{aligned} &|z_i| < 1 \qquad 2 \leqslant i \leqslant n \\ &|z_1 z_2 \ldots z_n| < 1. \end{aligned}$$

Since, as shown earlier that I_n is a symmetric function of the variables $z_1, z_2, \ldots, z_n$, we see that the conditions for convergence may be given as $|z_i| < 1, 1 \leqslant i \leqslant n$. This is to be compared with the more restrictive condition which is equivalent to $|z_i| < 1/2, 1 \leqslant i \leqslant n$ as derived in [1].

It is perhaps worthwhile mentioning in passing that the forms of the Laguerre polynomials in (1) are actually the Charlier polynomials $c_n(m; x)$, since [5]

$$\begin{aligned} &c_n(m; x) = (-x)^{-n} n! L_n^{(m-n)}(x) \\ &c_n(m; x) = c_m(n; x). \end{aligned}$$

From this observation, the multiple generating function in (1) may be written as

$$\sum_{s_1, s_2, \ldots, s_n = 0}^{\infty} \frac{(-z_1 x)^{s_1}}{s_1!} \frac{(-z_2 x)^{s_2}}{s_2!} \cdots \frac{(-z_n x)^{s_n}}{s_n!} c_{s_1}(s_2; x) c_{s_2}(s_3; x) \ldots c_{s_n}(s_1; x). \tag{14}$$

The following is Meixner's bilinear generating function for the Charlier polynomials [6], which is equivalent to the special case of the addition theorem of Vilenkin, viz

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} c_k(m; x) c_k(n; y) = \mathrm{e}^z \left(1 - \frac{z}{x}\right)^m \left(1 - \frac{z}{y}\right)^n c_m\left(n; -\frac{(x-z)(y-z)}{z}\right). \tag{15}$$

We could have used the formula in (15) to affect the successive reduction of the multiple generating function in (14), very much akin to the procedure we used earlier. We find, however, that in this form the recursion relations are somewhat unwieldy to use in order to arrive at the final solution.

References

[1] Messina A and Paladimo E 1996 *J. Phys. A: Math. Gen.* **29** L263

[2] Vilenkin N J 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society) p 433

[3] Miller W Jr 1970 *SIAM J. Math. Anal.* **1** 405

[4] Rainvillve E D 1960 *Special Functions* (New York: McMillan) p 213

[5] Erdélyi A, Magnus F, Oberhettinger F and Tricomi F G 1954 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) p 226

[6] Meixner J 1939 *Math. Zeitschr.* **44** 531